# First-order nonequilibrium phase transition in a spatially extended system

R. Müller, K. Lippert, A. Kühnel, and U. Behn
*Institut für Theoretische Physik, Universität Leipzig, Augustusplatz 10, D-04109 Leipzig, Germany*
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We investigate a system of harmonically coupled identical nonlinear constituents subject to noise in different spatial arrangements. For global coupling, we find for infinitely many constituents the coexistence of several ergodic components and a bifurcation behavior like in *first-order* phase transitions. These results are compared with simulations for finite systems both for global coupling and for nearest-neighbor coupling on two- and three-dimensional cubic lattices. The mean-field-type results for global coupling provide a better understanding of the more complex behavior in the latter case.
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The influence of noise on nonlinear systems is the subject of intense experimental and theoretical investigations [1]. Zero-dimensional models considering stochastic differential equations for a macroscopic order parameter homogeneous in space and coupling in a multiplicative way to the noise exhibit noise-induced transitions such as transitions between unimodal and bimodal stationary distributions [2,3]; cf. also [4] and references therein. Multiplicative noise is found in many different fields [1,4] including, e.g., directed percolation [5]; see also [6].

Real systems such as solids and liquid crystals are characterized by interactions between spatially distributed constituents. Spatially extended noisy systems described by stochastic *partial* differential equations are difficult to treat analytically; for recent studies see, e.g., [7,8]. Simulations, though expensive, may provide some guide to a theoretical understanding of those systems [9–13].

Models with global coupling of nonlinear noisy constituents are by far easier to investigate and allow even for analytical results [9,10,14–17]. For example, Shiino [16] was able to extend the concept of phase transitions to nonequilibrium phenomena described by globally coupled nonlinear oscillators subject to additive noise. More recently, Van den Broeck *et al.* [9,10] demonstrated the appearance of a *second-order* noise-induced phase transition in a model with globally coupled nonlinear constituents subject to multiplicative and additive noise, which shows no transitions in the absence of noise. In this paper we present a model constructed in a spirit similar to [9] that exhibits a *first-order* noise-induced phase transition connected with a hard onset of the coexisting ergodic components of the system. Varying parameters of the system or of the noise, the order of the phase transition may change, as we observed previously for zero-dimensional models [18].

We investigate a system of harmonically coupled identical nonlinear constituents under the influence of noise acting simultaneously in additive and multiplicative ways. We consider two cases distinguished by the spatial arrangement of the $L$ constituents: the case of global coupling of all components and the case of nearest-neighbor coupling on a $d$-dimensional cubic lattice. In the case of global coupling, analytic results are obtained for $L\to\infty$ and are compared with simulations for $L=100$ and $L=1000$, respectively. Furthermore, simulations were carried out for the case of nearest-neighbor coupling in $d=2$ ($L=100\times 100$) and $d=3$ ($L=20\times 20\times 20$). In these cases the results for global coupling can be considered as mean-field approximation.

The variables $x_i$ of the individual constituents at the lattice sites $i$ obey the following stochastic differential equations in the Stratonovich sense:

$$\dot{x}_i = f(x_i) + g(x_i)\xi_i - \frac{D}{N}\sum_{j\in\mathcal{N}(i)} (x_i - x_j). \tag{1}$$

Here $\mathcal{N}(i)$ denotes the set of involved neighbors of site $i$. The number of involved neighbors $N$ is equal to $L-1$ in the case of global coupling and to $2d$ in the case of nearest-neighbor coupling. The parameter $D$ controls the strength of the spatial interactions. The $\xi_i(t)$ represent zero mean spatially uncorrelated Gaussian white noise at point $i$ with the autocorrelation function

$$\langle \xi_i(t)\xi_j(t')\rangle = \sigma^2 \delta_{ij}\delta(t-t'), \tag{2}$$

where $\sigma^2$ is the noise strength. For nearest-neighbor coupling and suitable chosen parameters, Eq. (1) can be considered as a discretized version of a stochastic partial differential equation with diffusive coupling.

The stationary Fokker-Planck equation for the probability density of $x_i$ reads [9]

$$0 = \frac{\partial}{\partial x_i}\left( -f(x_i) + \frac{D}{N}\sum_{j\in\mathcal{N}(i)} (x_i - \langle x_j|x_i\rangle) + \frac{\sigma^2}{2} g(x_i)\frac{\partial}{\partial x_i} g(x_i)\right) P_s(x_i), \tag{3}$$

where $\langle x_j|x_i\rangle = \int dx_j x_j P_s(x_j|x_i)$ is the steady-state conditional average of $x_j$, $j\in\mathcal{N}(i)$, given $x_i$ at site $i$.

For the case of global coupling, fluctuations disappear in the average $1/(L-1)\Sigma_{j\in\mathcal{N}(i)}\langle x_j|x_i\rangle$ if $L\to\infty$. Considering the class of solutions for which this expression is independent of lattice site $i$, we can replace it by the steady-state mean value $\langle x\rangle$, which is self-consistently determined by

$$\langle x\rangle = \int_{-\infty}^{\infty} dx\; x P_s(x,\langle x\rangle) \equiv F(\langle x\rangle). \tag{4}$$

 

Obviously, for finite lattices or coupling to a finite subset of neighbors this replacement represents a mean-field approximation. Following Shiino [16], one obtains the same results by replacing in Eq. (1) the spatial average $(1/N)\Sigma_{j\in\mathcal{N}(i)}x_j$ by the statistical average $\langle x\rangle$.

In this paper we consider a simple model, for which a nontrivial solution of Eq. (4) is not emerging from zero but appears with a jump to a nonzero value at a critical value of the control parameter. The model is specified by

$$f(x)=ax+x^3-x^5, \quad g(x)=1+x^2. \tag{5}$$

For the model with global coupling the stationary probability density is

$$P_s(x,\langle x\rangle)\propto(1+x^2)^{3/\sigma^2-1}\exp\left\{\sigma^{-2}\left[-x^2+\frac{D-a+2}{1+x^2}+D\langle x\rangle\left(\frac{x}{1+x^2}+\arctan(x)\right)\right]\right\}. \tag{6}$$

Without spatial coupling ($D=0$) the model shows the following bifurcation behavior. In the deterministic case ($\sigma^2=0$) the stationary solution undergoes a subcritical bifurcation at $a_c=0$. The multiplicative noise shifts the bifurcation threshold of the maximum of the stationary probability density to $a_c=\sigma^2$. For weak noise ($\sigma^2<1$) the bifurcation is subcritical, whereas for $\sigma^2\geqslant1$ it is supercritical.

In the following we mainly restrict ourselves, for the sake of convenience, to the case $\sigma^2=1$, where the noise intensity is just sufficient to produce a change from the deterministic subcritical bifurcation into a supercritical bifurcation. Then the extreme values of the stationary probability density are $x_{st}=0$ and, if $a>1$, $x_{st}=\pm(a-1)^{1/4}$.

The global coupling ($D\neq0$) favors a coherent behavior of the spatially distributed components, which is, in a sense, an effect opposite to the noise and will ''restore'' the subcritical bifurcation. Hence we expect to observe a first-order nonequilibrium phase transition.

The bifurcation behavior of $\langle x\rangle$ is governed by the self-consistency condition (4). Since for our model $F(\langle x\rangle)$ is an odd function of $\langle x\rangle$ we always have the solution $\langle x\rangle=0$. Moreover, pairs of new stable and unstable nonzero solutions may occur in certain parameter ranges. Only stable solutions can be observed in simulations. We remark without proof that $F'(\langle x\rangle)<1$ is sufficient for stability (cf. [16]). The existence of more than one stable solution leads to the existence of several corresponding stationary probability densities $P_s(x,\langle x\rangle)$. Therefore, a phase transition breaking the ergodicity of the system is expected in the case of global coupling.

The typical behavior of $F(\langle x\rangle)$ for our model is sketched in Fig. 1. Whereas the model investigated by Van den Broeck *et al.* [9] exhibits a *second-order* phase transition, we find a *first-order* phase transition connected with a *hard* onset of the nontrivial stable solution of the self-consistency condition for our model in a certain parameter range where the model without spatial coupling exhibits only the trivial solution $\langle x\rangle=0$.

The phase diagram given in Fig. 2 confirms the intuitive picture drawn above: The spatial coupling favors coherent behavior of the components, acting thus oppositely to the noise. The critical value of $a$ is reduced with increasing coupling strength $D$ and above a critical strength of $D$ the transition is of first order.

Figure 3 shows the different solutions of the self-consistency equation (4) for the order parameter $\langle x\rangle$ leading to different ergodic components of the system as a function of the parameter $a$, the spatial coupling constant $D$, and the noise strength $\sigma^2$, respectively. In all cases a hard onset of the nontrivial stable solutions can be observed.

It is instructive to compare the results obtained by solution of the self-consistency condition (4) with simulations on

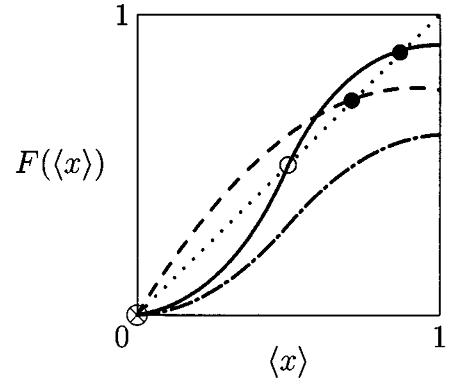


FIG. 1. Solutions of the self-consistency equation (4), $F(\langle x\rangle)=\langle x\rangle$, in three typical cases. $\langle x\rangle=0$ is always a solution; in the case of the dash-dotted line it is the only solution. In the case considered by Van den Broeck *et al.* [9] (dashed line) we have two stable solutions $\langle x\rangle=\pm x_s$ (full circle) and $\langle x\rangle=0$ is unstable. In the case considered here (solid line) we have in addition to the stable solution $\langle x\rangle=0$ a pair of unstable solutions $\langle x\rangle=\pm x_u$ (empty circle) and a pair of stable solutions $\langle x\rangle=\pm x_s$ (full circle). In contrast to the former case, in the latter case the nontrivial solutions do not emerge continuously from $\langle x\rangle=0$ but appear with nonzero value at the critical value of the control parameter. This indicates a *first-order* nonequilibrium phase transition.

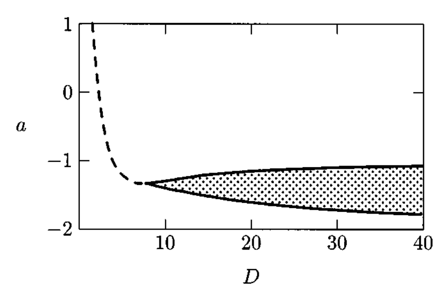


FIG. 2. Phase diagram in the case of global coupling for $\sigma^2=1$. For small $D$ we have a second-order transition. The spatial coupling favors a coherent behavior of the constituents, acting thus oppositely to the noise. With increasing coupling strength $D$ the critical value of $a$ is reduced and above a critical strength of $D$ the first-order transition of the model without noise and spatial coupling is ''restored.'' The solid and dashed lines denote first- and second-order nonequilibrium phase transitions, respectively. The number of ergodic components is 3 in the shadowed region, 2 in the region above, and 1 in the region below. Hysteresis appears in the shadowed region.

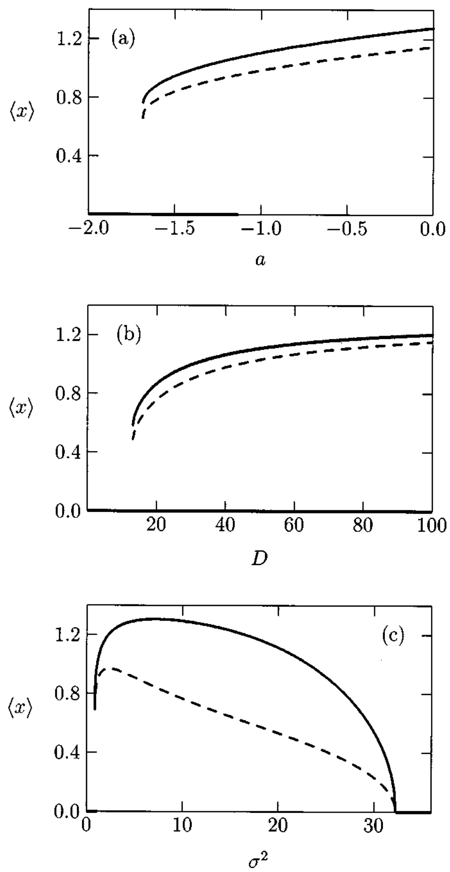


FIG. 3. Stable solutions of the self-consistency equation (4) as a function of the control parameters (solid lines) determine the ergodic components. Unstable solutions (not shown here) cannot be observed in simulations. The maxima of the stationary probability density (dashed line) of the corresponding ergodic components (6) exhibit qualitatively the same behavior. In all the diagrams we observe at the critical value a *hard* onset of the nontrivial stable solution corresponding to a *first-order* transition. In (c) a reentrant behavior is found similar to that in [9]. The parameters are (a) $\sigma^2=1$ and $D=25$, (b) $\sigma^2=1$ and $a=-1.5$, and (c) $a=-1.5$ and $D=25$.

*finite* globally coupled systems of different size. We consider a parameter setting where three different stable solutions of Eq. (4) exist. For small systems ($L=100$) the ergodicity breaking is not perfect. We still observe a few transitions between the different ''ergodic components'' due to large fluctuations of $\langle x(t)\rangle$ around its stationary values. The trajectory of the spatial average $\langle x(t)\rangle_L=(1/L)\Sigma_{i=1}^{L}x_i(t)$ is shown in Fig. 4. For larger systems ($L=1000$) the fluctuations become smaller and the system remains very long inside one of the ergodic components. In that case, there are practically no transitions. The initial conditions determine which of the ergodic components is selected.

We also performed simulations of the stochastic differential equation (1) for nearest-neighbor coupling on a three-dimensional cubic lattice with $L=20\times20\times20$ sites and on a two-dimensional square lattice with $L=100\times100$ sites. Qualitatively, one gets a behavior very similar to the case of global coupling. In Fig. 5 we compare the probability density $P_s(x)$ at one lattice site given by Eq. (6) for the case of global coupling with the results of simulations for nearest-neighbor coupling on the three-dimensional lattice. Although

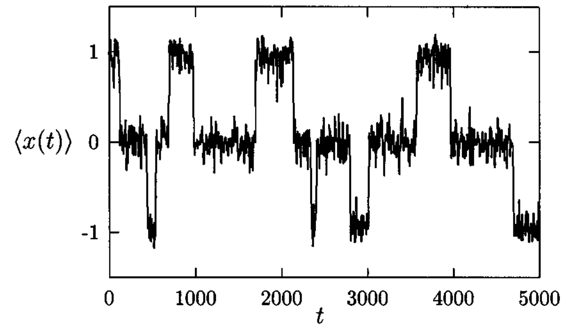


FIG. 4. Trajectory of the spatial average $\langle x(t)\rangle_L=(1/L)\Sigma_{i=1}^{L}x_i(t)$ for the case of global coupling ($L=100$, $\sigma^2=1$, $a=-1.48$, and $D=30$). The trajectory fluctuates preferably around the mean values $x_s=0$ and $x_s=\pm0.94$; sometimes large fluctuations lead to jumps between the ''ergodic components.''

for the finite system there is no *perfect* separation into different ergodic components, the trajectories remain very long in the corresponding ''basin of attraction.'' The histograms obtained by sampling those trajectories follow very closely the probability densities of the ergodic components for global coupling. The value of $x_s$ for nearest-neighbor coupling is about 10% smaller than for global coupling.

Simulations on the two-dimensional square lattice exhibit a similar qualitative behavior; the quantitative agreement with the results of the globally coupled model is, as expected, less satisfactory. Figure 6 compares the order parameter as a function of $a$ and $\sigma^2$ with the results for global coupling. Although the bistable region is smaller than in the case of global coupling, it no doubt exists. We remark that the fluctuations (indicated by the error bars in Fig. 6) are larger for the states with $\langle x\rangle\neq0$ than those for $\langle x\rangle=0$, being a clear indication of the multiplicative nature of the driving process. As in the case of global coupling, for the $L=100\times100$ system the trajectory stays inside the ergodic component selected by the initial condition for a very long

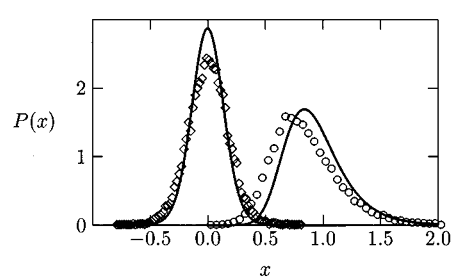


FIG. 5. Probability densities for $x_i(t)$ at arbitrary $i$ for the case of global coupling (solid lines) as given by Eq. (6). The ergodic components correspond to $\langle x\rangle=0$, and $x_s=0.94$ ($\sigma^2=1$, $D=25$, and $a=-1.5$). These results are compared with simulations for the three-dimensional cubic lattice ($L=20\times20\times20$) with nearest-neighbor coupling. The probability densities (◇ and ○) are obtained by sampling 20 000 equidistant points within a trajectory of length $t=10\,000$ near $\langle x\rangle=0$ and $x_s=0.85$ separately. The plot indicates that the globally coupled system gives a good idea of the qualitative and quantitative behavior of the system with nearest-neighbor coupling.

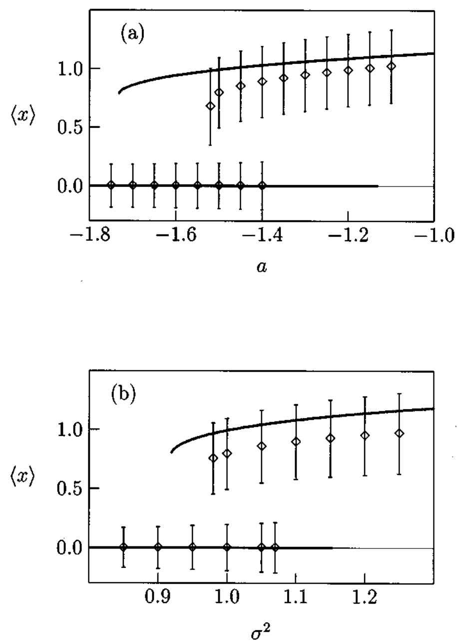


FIG. 6. Comparison of the order parameter $\langle x\rangle$ obtained by simulation for a two-dimensional square lattice of size $L=100\times 100$ with the results for the globally coupled model (thick solid line) for $D=30$. The diamonds denote the average of $x_i(t)$ over all lattice sites and over a time span of order 100 during which no jumps between the ergodic components occur. The error bars indicate the time average over the standard deviation $\{(1/L)\Sigma_i[x_i(t)-\langle x(t)\rangle_L]^2\}^{1/2}$. (a) and (b) show the dependence on the control parameter $a$ ($\sigma^2=1$) and the noise strength $\sigma^2$ ($a=-1.5$), respectively. The coexistence of the solutions with $\langle x\rangle\neq 0$ and $\langle x\rangle=0$ over a range of parameters as for the case of global coupling is obvious.

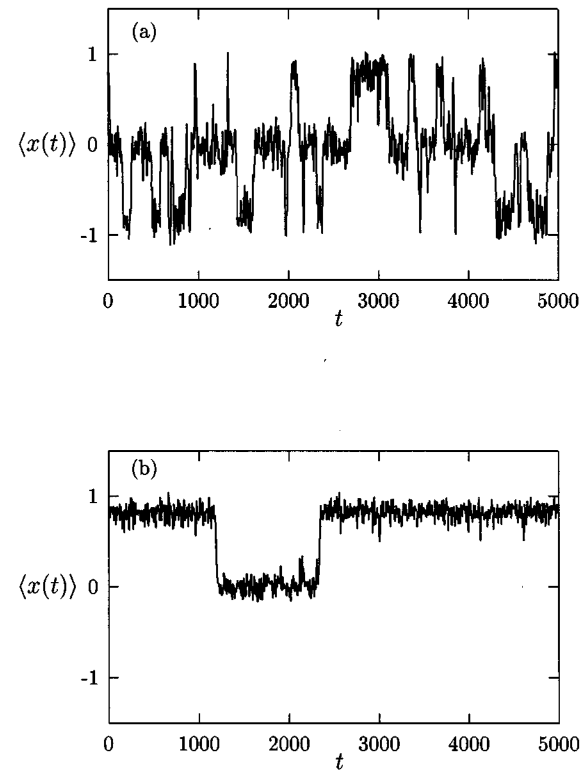


FIG. 7. Trajectories of the spatial average $\langle x(t)\rangle_L$ for the two-dimensional square lattice of size (a) $L=10\times 10$ and (b) $L=18\times 18$ for the same parameters as in Fig. 4. In the smaller system frequent jumps between the ergodic components are observed; with increasing size of the system these events are rarefied. Already for a size of $L=100\times 100$ no jumps were observed in time spans of order 5000.

time. No jumps were observed in our simulations running typically over a time $t=5000$. Jumps between the ergodic components induced by large fluctuations are observed in simulations of smaller systems. With increasing size they become less frequent; cf. Fig. 7.

In this paper we investigated a model that exhibits a *first-order* nonequilibrium phase transition due to a hard onset of the coexistence of several stable ergodic components of the system. Other models that we investigate at present exhibit the same behavior. We also found changes from second-order to first-order transitions by tuning parameters of the noise or the system. In any case, both the nonlinear terms and the interplay between deterministic and stochastic effects determine the order of the transition. In previous work [9–13] only second-order noise-induced nonequilibrium phase transitions have been observed. In a different context, a system of coupled Duffing oscillators was used to describe a liquid to crystal transition [19]. For this model mean-field theory yields a first-order nonequilibrium transition, which is preserved including *additive* noise.

Our results may be of interest in the context of experimental investigations in electrohydrodynamic convection in nematic liquid crystals subject to thermal fluctuations (additive noise) and/or an external stochastic voltage (multiplicative noise). There are experimental hints [20] that the first transition from the homogeneous state to the structured state might be weakly hysteretic although, the deterministic theory predicts a supercritical bifurcation.

*Note added*. After completion of this work we got knowledge of a paper by S. Kim, S. H. Park, and C. S. Ryn [Phys. Rev. Lett. **78**, 1616 (1997)] where a first-order noise-induced transition on a different system of globally coupled oscillators is described.

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